

A QUESTION ON SPLITTING OF METAPLECTIC COVERS

SHIV PRAKASH PATEL

ABSTRACT. Let E/F be a quadratic extension of a non-Archimedean local field. Splitting of the 2-fold metaplectic cover of $\mathrm{Sp}_{2n}(F)$ when restricted to various subgroups of $\mathrm{Sp}_{2n}(F)$ plays an important role in application of the Weil representation of the metaplectic group. In this paper we prove the splitting of the metaplectic cover of $\mathrm{GL}_2(E)$ over the subgroups $\mathrm{GL}_2(F)$ and D_F^\times , where D_F is the quaternion division algebra with center F , as a first step in our study of the restriction of representations of metaplectic cover of $\mathrm{GL}_2(E)$ to $\mathrm{GL}_2(F)$ and D_F^\times . These results were suggested to the author by Professor Dipendra Prasad.

1. INTRODUCTION

This paper will be concerned with certain 2-fold covers of $\mathrm{GL}_2(E)$, where E is a non-Archimedean local field to be called the metaplectic covering of $\mathrm{GL}_2(E)$. We recall that there is a unique (up to isomorphism) non-trivial 2-fold cover of $\mathrm{SL}_2(E)$ called the metaplectic cover and denoted by $\widetilde{\mathrm{SL}}_2(E)$ in this paper, but there are many inequivalent 2-fold coverings of $\mathrm{GL}_2(E)$ which extend this 2-fold covering of $\mathrm{SL}_2(E)$. We fix a covering of $\mathrm{GL}_2(E)$ as follows. Observe that $\mathrm{GL}_2(E)$ is the semi-direct product of $\mathrm{SL}_2(E)$ and E^\times , where E^\times sits inside $\mathrm{GL}_2(E)$ as $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$. This action of E^\times on $\mathrm{SL}_2(E)$ lifts uniquely to an action of E^\times on $\widetilde{\mathrm{SL}}_2(E)$. Denote $\widetilde{\mathrm{GL}}_2(E) = \widetilde{\mathrm{SL}}_2(E) \rtimes E^\times$ and call this the metaplectic cover of $\mathrm{GL}_2(E)$. Thus the metaplectic cover of $\mathrm{GL}_2(E)$ that we consider in this paper is that cover of $\mathrm{GL}_2(E)$ which extends the metaplectic cover of $\mathrm{SL}_2(E)$ and is further split on the subgroup $\left\{ \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} : e \in E^\times \right\}$.

Given a central extension of a group G by $\mathbb{Z}/2\mathbb{Z}$ say

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G' \longrightarrow G \longrightarrow 1$$

there is a natural central extension, say G'' , of G by \mathbb{C}^\times , given by

$$G'' := G' \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}^\times := \frac{G' \times \mathbb{C}^\times}{\langle (-1, -1) \rangle}$$

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which sits in the following exact sequence

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & G' & \longrightarrow & G \longrightarrow \{1\} \\ & & \downarrow & & \downarrow & & \parallel \\ \{1\} & \longrightarrow & \mathbb{C}^\times & \longrightarrow & G'' & \longrightarrow & G \longrightarrow \{1\} \end{array}$$

This \mathbb{C}^\times -central extension of G is said to be obtained from the 2-fold cover of G . It is well known that \mathbb{C}^\times -covers tend to be easier to analyse and which is what we shall do in this paper.

Let F be a non-Archimedean local field of characteristic zero and E is a quadratic extension of F . Let D_F denote the unique quaternion division algebra with center F . Note that $D_F^\times \hookrightarrow \mathrm{GL}_2(E)$ given by fixing an isomorphism $D_F \otimes E \cong M_2(E)$. By Skolem-Noether theorem, such an embedding is uniquely determined upto conjugation by elements of $\mathrm{GL}_2(E)$. The main theorem of this paper is the following:

Theorem 1.1. *Let E be a quadratic extension of a non-Archimedean local field and $\widetilde{\mathrm{GL}}_2(E)$ be the two-fold metaplectic covering of $\mathrm{GL}_2(E)$. Then:*

- (1) *The two-fold metaplectic covering splits over the subgroup $\mathrm{GL}_2(F)$.*
- (2) *The \mathbb{C}^\times -covering obtained from $\widetilde{\mathrm{GL}}_2(E)$ splits over the subgroup D_F^\times .*

Note that a quadratic extension L of F gives rise to two embeddings of L in $M(2, E)$ as in the diagram below:

$$\begin{array}{ccc} & D_F & \\ \nearrow & & \searrow \\ L & & M(2, E) \\ \searrow & & \nearrow \\ & M(2, F) & \end{array}$$

By Skolem-Noether theorem, any two embeddings of $L \otimes E$ in $M(2, E)$ and hence of L are conjugate in $M(2, E)$ by $\mathrm{GL}_2(E)$.

The following refined question was formulated by Dipendra Prasad.

Refined Question 1.2. *Does there exist a natural identification of the set of splittings of the \mathbb{C}^\times -cover of $\mathrm{GL}_2(E)$ restricted to $\mathrm{GL}_2(F)$ and set of splittings restricted to D_F^\times (in either of the two cases the set of splittings is a principal homogeneous space over the character group of F^\times) such that for any quadratic extension L of F , the two embeddings*

of L^\times in the \mathbb{C}^\times -covers $\widetilde{\mathrm{GL}}_2(E)$ (which we take to be \mathbb{C}^\times -central extension of $\mathrm{GL}_2(E)$)

$$\begin{array}{ccc}
 & D_F^\times & \\
 \nearrow & & \searrow j \\
 L^\times & & \widetilde{\mathrm{GL}}_2(E) \\
 \searrow & & \nearrow i \\
 & \mathrm{GL}_2(F) &
 \end{array}$$

are conjugate in the \mathbb{C}^\times cover $\widetilde{\mathrm{GL}}_2(E)$?

We are not able to handle the refined question, and will only contend with the proof of the splitting of the metaplectic cover of $\mathrm{GL}_2(E)$ restricted to D_F^\times . However the above refined question plays an important role in harmonic analysis relating the pair $(\widetilde{\mathrm{GL}}_2(E), \mathrm{GL}_2(F))$ with the pair $(\widetilde{\mathrm{GL}}_2(E), D_F^\times)$.

We briefly say a few words about the proofs. The proof for $\mathrm{GL}_2(F)$ is straight forward from the explicit knowledge of the cocycle defining the metaplectic cover. For any quadratic extension L of F , we know that the embedding $L^\times \hookrightarrow D_F^\times$ is conjugate inside $\mathrm{GL}_2(E)$ to the embedding of L^\times inside $\mathrm{GL}_2(E)$ realized as $L^\times \hookrightarrow \mathrm{GL}_2(F) \hookrightarrow \mathrm{GL}_2(E)$. Since the metaplectic cover of $\mathrm{GL}_2(E)$ is split over $\mathrm{GL}_2(F)$, it is split in particular over L^\times for any quadratic extension L of F . Thus we know that the restriction of the metaplectic cover of $\mathrm{GL}_2(E)$ to D_F^\times has the property that it splits over L^\times of any quadratic extension L of F . This is the key property to be used in the proofs below. The coverings considered in this paper are topological coverings which are locally split, i.e. all the coverings considered here split when restricted to a small neighbourhood of identity. Although it appears that we deal with abstract coverings we do not emphasize every time on the fact that these are topological coverings.

2. SPLITTING OVER $\mathrm{GL}_2(F)$

We prove the following proposition:

Proposition 2.1. *Let E be a quadratic extension of a non-Archimedean local field F . Then the metaplectic 2-fold cover $\widetilde{\mathrm{GL}}_2(E)$ of $\mathrm{GL}_2(E)$, as described in the introduction, splits over the subgroup $\mathrm{GL}_2(F)$.*

Proof. To prove that the covering $\widetilde{\mathrm{GL}}_2(E)$ of $\mathrm{GL}_2(E)$ splits over $\mathrm{GL}_2(F)$, we observe that the two-cocycle β which defines the two-fold metaplectic cover satisfies $\beta(\sigma, \tau) = 1$ for all $\sigma, \tau \in \mathrm{GL}_2(F)$, i.e., the cocycle is identically 1 when restricted to $\mathrm{GL}_2(F)$. One knows that the defining expression of the cocycle β involves only quadratic Hilbert symbols of the field E . The proof will follow if we prove that restriction of the quadratic Hilbert symbol of E restricted to F is identically 1, which is the content of the next lemma. \square

Lemma 2.2. *If we denote the quadratic Hilbert symbol of the field E by $(\cdot, \cdot)_E$. Then*

$$(a, b)_E = 1 \text{ for all } a, b \in F^\times.$$

Proof. Let $(\cdot, \cdot)_F$ denotes the quadratic Hilbert symbol of the field F . Then it is well known that for $a \in F^\times$ and $b \in E^\times$, we have

$$(a, b)_E = (a, Nb)_F.$$

Hence for $a, b \in F^\times$ we have

$$(a, b)_E = (a, Nb)_F = (a, b^2)_F = 1. \quad \square$$

3. SPLITTING OVER $\mathrm{SL}_1(D_F)$

Recall that D_F denotes the unique quaternion division algebra over the field F and $\mathrm{SL}_1(D_F)$ is the subgroup of norm 1 elements. Fix an embedding $E \hookrightarrow D_F$ through which D_F can be realized as a two dimensional vector space over E with E acting on D_F on the left and D_F acting on itself on the right. This gives rise to an embedding $D_F^\times \hookrightarrow \mathrm{GL}_2(E)$. Since $\mathrm{SL}_1(D_F)$ is compact, we can assume that $\mathrm{SL}_1(D_F) \subset \mathrm{GL}_2(\mathcal{O}_E)$. It is well known that if the residue characteristic of F is odd, then the two-fold metaplectic cover $\widetilde{\mathrm{GL}}_2(E)$ of $\mathrm{GL}_2(E)$ splits over $\mathrm{GL}_2(\mathcal{O}_E)$ and hence over $\mathrm{SL}_1(D_F)$. Such a simple minded proof does not work for $p = 2$, however, we prove in this section that the \mathbb{C}^\times metaplectic cover of $\mathrm{SL}_2(E)$ splits when restricted to $\mathrm{SL}_1(D_F)$.

Proposition 3.1. *Restriction of the non-trivial 2-fold cover of $\mathrm{SL}_4(F)$ to $\mathrm{SL}_2(E)$ remains non-trivial, hence gives the unique non-trivial 2-fold cover of $\mathrm{SL}_2(E)$.*

Proof. The proposition amounts to the assertion that there is a commutative diagram between the unique 2-fold covers of $\mathrm{SL}_2(E)$ and $\mathrm{SL}_4(F)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{\mathrm{SL}}_2(E) & \longrightarrow & \mathrm{SL}_2(E) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{\mathrm{SL}}_4(F) & \longrightarrow & \mathrm{SL}_4(F) \longrightarrow 1 \end{array}$$

This follows from the generality that the transfer map from $K_2(E)/2K_2(E)$ to $K_2(F)/2K_2(F)$ is an isomorphism [Mil71]; we omit the details. \square

Corollary 3.2. *The \mathbb{C}^\times -cover of $\mathrm{SL}_2(E)$ obtained from $\widetilde{\mathrm{SL}}_2(E)$ splits over $\mathrm{SL}_1(D_F)$.*

Proof. From the proposition 3.1, restriction of the 2-fold cover from $\mathrm{SL}_4(F)$ to $\mathrm{SL}_2(E)$ remains non-trivial. Since we have inclusion of groups

$$\mathrm{SL}_2(E) \hookrightarrow \mathrm{Sp}_4(F) \hookrightarrow \mathrm{SL}_4(F),$$

and all these groups have a unique non-trivial 2-fold cover, we deduce that the unique non-trivial 2-fold cover of $\mathrm{SL}_4(F)$ restricts to give the unique non-trivial 2-fold cover of

$\mathrm{Sp}_4(F)$ which in turn restricts to the unique non-trivial 2-fold cover of $\mathrm{SL}_2(E)$. Now we use the inclusion of the groups

$$\mathrm{SL}_1(D_F) \hookrightarrow \mathrm{SL}_2(E) \hookrightarrow \mathrm{Sp}_4(F)$$

and use a result of Kudla [Kud94, Theorem 3.1] according to which the restriction of the \mathbb{C}^\times -covering of $\mathrm{Sp}_4(F)$ to $\mathrm{U}(2)$ splits. (The result of Kudla is valid for any unitary group $\mathrm{U}(n)$ defined by a skew hermitian form in n variables over E which comes with a natural embedding in $\mathrm{Sp}_{2n}(F)$). If we take the hermitian form in 2 variables which is anisotropic, then for the corresponding unitary group $\mathrm{U}(2)$, $\mathrm{SU}(2) \cong \mathrm{SL}_1(D_F)$. As a result, restriction of the \mathbb{C}^\times -covering from $\mathrm{SL}_2(E)$ to $\mathrm{SL}_1(D) = S\mathrm{U}(2) \subset \mathrm{U}(2)$ splits. \square

4. SPLITTING OVER D_F^\times

In this section we prove the splitting of \mathbb{C}^\times -cover of $\mathrm{GL}_2(E)$ obtained from $\widetilde{\mathrm{GL}_2(E)}$ when restricted to D_F^\times .

4.1. Even residue characteristic case. Note the following short exact sequence

$$1 \longrightarrow \mathrm{SL}_1(D_F) \longrightarrow D_F^\times \longrightarrow F^\times \longrightarrow 1. \quad (\text{A})$$

Let \mathbb{C}^\times be the trivial D_F^\times -module. Then $H^2(D_F^\times, \mathbb{C}^\times)$ classifies central extensions of D_F^\times by the group \mathbb{C}^\times . The Hochschild-Serre spectral sequence arising from (A) gives a filtration on $H^2(D_F^\times, \mathbb{C}^\times)$:

$$H^2(D_F^\times, \mathbb{C}^\times) = F^0 \supseteq F^1 \supseteq F^2 \supseteq 0$$

with $F^0/F^1 = E_\infty^{0,2}$, $F^1/F^2 = E_\infty^{1,1}$ and $F^2 = E_\infty^{2,0}$, where

$$\begin{aligned} E_2^{0,2} &= H^0(F^\times, H^2(\mathrm{SL}_1(D_F), \mathbb{C}^\times)), \\ E_2^{1,1} &= H^1(F^\times, H^1(\mathrm{SL}_1(D_F), \mathbb{C}^\times)), \\ E_2^{2,0} &= H^2(F^\times, H^0(\mathrm{SL}_1(D_F), \mathbb{C}^\times)). \end{aligned}$$

Consider the embedding $D_F^\times \hookrightarrow \mathrm{GL}_2(E)$ and denote the restriction of the central extension of $\mathrm{GL}_2(E)$ to D_F^\times as well as the corresponding element of $H^2(D_F^\times, \mathbb{C}^\times)$ by β . In the section 3 we proved that β restricted to $\mathrm{SL}_1(D_F)$ is trivial, therefore $\beta \in F^1$. In even residue characteristic, since we are dealing with a cohomology class of order 2 (or 1), the following result of C. Riehm [Rie70] implies that β must be trivial in F^1/F^2 .

Proposition 4.1 (Riehm). *Let $G_0 = \mathrm{SL}_1(D_F)$ and G_i for $i \geq 1$ be the standard congruence subgroup of G_0 . The*

$$[G_0, G_0] = G_1.$$

In particular, the character group of $\mathrm{SL}_1(D_F)$ is a finite cyclic group of order prime to p .

Thus in the even residue characteristic an element of $H^2(D_F^\times, \mathbb{C}^\times)$ of order 2 which is trivial when restricted to $\mathrm{SL}_1(D_F)$ arises from the inflation of an element of $H^2(F^\times, \mathbb{C}^\times)$. An element of $H^2(F^\times, \mathbb{C}^\times)$ is represented by a central extension

$$1 \rightarrow \mathbb{C}^\times \rightarrow \widetilde{F^\times} \rightarrow F^\times \rightarrow 1.$$

The proof of the splitting of the \mathbb{C}^\times -metaplectic cover of $\mathrm{GL}_2(E)$ restricted to D_F^\times will be completed in even residue characteristic once we prove the following lemma.

Lemma 4.2. *A \mathbb{C}^\times -covering of D_F^\times coming from a \mathbb{C}^\times covering of F^\times via the norm map, which is trivial on L^\times for all quadratic extensions L of F , is trivial.*

Proof. Suppose that there exists a non-trivial \mathbb{C}^\times -covering of D_F^\times coming from a \mathbb{C}^\times -central extension \widetilde{F}^\times of F^\times via the norm map, which is trivial on L^\times for all quadratic extension L of F . Thus there are two elements $e_1, e_2 \in \widetilde{F}^\times$ which do not commute. Look at the images, say, f_1, f_2 of e_1, e_2 in F^\times . Let \bar{f}_1, \bar{f}_2 be images of f_1, f_2 in $F^\times/F^{\times 2}$. Since residue characteristic of F is even, $F^\times/F^{\times 2}$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension ≥ 3 . Therefore given any two elements $\bar{f}_1, \bar{f}_2 \in F^\times/F^{\times 2}$, there exist a subgroup $F_1 \hookrightarrow F^\times$ of index 2 containing f_1, f_2 . By local class field theory, there exists a unique quadratic extension M of F with $\mathrm{Norm}_{M/F}(M^\times) = F_1$. Now we use the fact given to us that the central extension of D_F^\times that we are considering is trivial on L^\times for any quadratic extension L of F , in particular on M^\times . Hence the inverse image of M^\times in the central extension must be abelian, a contradiction to the construction of M . \square

4.2. Odd residue characteristic case. In this subsection, we assume that the residue characteristic p of F is odd. We first introduce more notation. Let \mathcal{O}_{D_F} be the maximal compact subring of D_F and \mathcal{P}_{D_F} be the maximal ideal of \mathcal{O}_{D_F} . Let $D_F^\times(1) := 1 + \mathcal{P}_{D_F}$. Note that $D_F^\times(1)$ is a normal pro- p subgroup in D_F^\times . Since p is odd and $D_F^\times(1)$ is a normal pro- p subgroup

$$H^2(D_F^\times, \mathbb{Z}/2\mathbb{Z}) \cong H^2(D_F^\times/D_F^\times(1), \mathbb{Z}/2\mathbb{Z}).$$

In other words, every 2-fold central extension of D_F^\times arises as a pull back of a 2-fold central extension $D_F^\times/D_F^\times(1)$. The structure of the group $D_F^\times/D_F^\times(1)$ is $\mathbb{F}_{q^2}^\times \rtimes \mathbb{Z}$, where \mathbb{F}_{q^2} is the finite field with q^2 elements and \mathbb{Z} operates on $\mathbb{F}_{q^2}^\times$ by powers of the Frobenius map $x \mapsto x^q$. This group sits in the following short exact sequence

$$0 \rightarrow \mathbb{F}_{q^2}^\times \rightarrow G' := D_F^\times/D_F^\times(1) \rightarrow \mathbb{Z} \rightarrow 0.$$

Using this description of the group we prove the following proposition.

Proposition 4.3. (A) *We have*

$$H^2(D_F^\times, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

(B) *If we denote 2-torsion element of $H^2(D_F^\times, \mathbb{C}^\times)$ by $H^2(D_F^\times, \mathbb{C}^\times)[2]$ then*

$$H^2(D_F^\times, \mathbb{C}^\times)[2] = \mathbb{Z}/2\mathbb{Z}.$$

Proof. (A) Since $G' = \mathbb{F}_{q^2}^\times \rtimes \mathbb{Z}$ and \mathbb{Z} has cohomological dimension 1, the Hochschild-Serre spectral sequence $E_2^{i,j} = H^i(\mathbb{Z}, H^j(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z}))$ calculating the cohomology of G' has $E_2^{1,1} = E_\infty^{1,1}$, $E_2^{0,2} = E_\infty^{0,2}$ and $E_2^{2,0} = E_\infty^{2,0} = 0$. Therefore

$$0 \longrightarrow H^1(\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z})) \longrightarrow H^2(G', \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \longrightarrow 0.$$

Since $H^1(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H^2(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and since \mathbb{Z} must act trivially on $\mathbb{Z}/2\mathbb{Z}$, we get

$$0 \rightarrow H^1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(G', \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

which proves part (A) of the proposition.

(B) This is evident from the short exact sequence of next Lemma 4.4.

This proves the proposition. \square

By proposition 4.3 there are four non-isomorphic two-fold coverings of the group D_F^\times . The lemma below proves that one of these non-trivial 2-fold covers becomes trivial as a \mathbb{C}^\times -cover.

Lemma 4.4. *We have the following short exact sequence*

$$0 \rightarrow \frac{H^1(D_F^\times, \mathbb{C}^\times)}{2H^1(D_F^\times, \mathbb{C}^\times)} \rightarrow H^2(D_F^\times, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(D_F^\times, \mathbb{C}^\times)[2] \rightarrow 0$$

with

$$\frac{H^1(D_F^\times, \mathbb{C}^\times)}{2H^1(D_F^\times, \mathbb{C}^\times)} \cong \mathbb{Z}/2\mathbb{Z}$$

where for any abelian group A , $A[2] = \{a \in A : 2a = 0\}$.

Proof. The short exact sequence can be deduced from the long exact sequence of cohomology groups of D_F^\times arising from the following short exact sequence

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{C}^\times \xrightarrow{2} \mathbb{C}^\times \rightarrow 0.$$

Since $[D_F^\times, D_F^\times] = \text{SL}_1(D_F)$ and $D_F^\times/\text{SL}_1(D_F) \cong F^\times$, the second statement follows from the fact that the character group of D_F^\times , i.e. $H^1(D_F^\times, \mathbb{C}^\times)$, is the same as the character group of F^\times , and in the odd residue characteristic, it is easy to see that

$$\frac{H^1(F^\times, \mathbb{C}^\times)}{2H^1(F^\times, \mathbb{C}^\times)} \cong \mathbb{Z}/2\mathbb{Z}. \quad \square$$

Proposition 4.5. *Let M be the quadratic unramified extension of F with $M \hookrightarrow D_F$. Then a 2-fold cover of D_F^\times which remains non-trivial with \mathbb{C}^\times coefficients does not split over the subgroup $M^\times \hookrightarrow D_F^\times$.*

Proof. Let $M^\times(1) = 1 + \mathcal{P}_M$. As M is a quadratic unramified extension of F , we have

$$M^\times/M^\times(1) \cong \mathbb{F}_{q^2}^\times \times \mathbb{Z}.$$

Since \mathbb{Z} has cohomological dimension 1, by Kunneth theorem

$$\begin{aligned} H^2(M^\times/M^\times(1), \mathbb{Z}/2\mathbb{Z}) &\cong H^2(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z}) \oplus (H^1(\mathbb{F}_{q^2}^\times, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Since $M^\times/M^\times(1) \cong \mathbb{F}_{q^2}^\times \times \mathbb{Z}$, its character group is isomorphic to $\widehat{\mathbb{F}_{q^2}^\times} \times \mathbb{C}^\times$. So once again as in Lemma 4.4, we get the following short exact sequence:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} = \frac{H^1(M^\times/M^\times(1), \mathbb{C}^\times)}{2H^1(M^\times/M^\times(1), \mathbb{C}^\times)} \rightarrow H^2(M^\times, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(M^\times, \mathbb{C}^\times) \rightarrow 0$$

By considering the embedding $M^\times/M^\times(1) \hookrightarrow D_F^\times/D_F^\times(1) = G'$, we get the following exact sequences with connecting homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{H^1(G', \mathbb{C}^\times)}{2H^1(G', \mathbb{C}^\times)} & \longrightarrow & H^2(G', \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^2(G', \mathbb{C}^\times)[2] \longrightarrow 0 \quad (**) \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & \frac{H^1\left(\frac{M^\times}{M^\times(1)}, \mathbb{C}^\times\right)}{2H^1\left(\frac{M^\times}{M^\times(1)}, \mathbb{C}^\times\right)} & \longrightarrow & H^2\left(\frac{M^\times}{M^\times(1)}, \mathbb{Z}/2\mathbb{Z}\right) & \longrightarrow & H^2\left(\frac{M^\times}{M^\times(1)}, \mathbb{C}^\times\right)[2] \longrightarrow 0 \end{array}$$

In the next lemma we prove that h is injective. This proves the proposition. \square

Lemma 4.6. *The right most vertical map $h : H^2(G', \mathbb{C}^\times)[2] \rightarrow H^2(M^\times, \mathbb{C}^\times)[2]$ in the above diagram ** is an isomorphism.*

Proof. Consider the short exact sequence which appeared in the proof of Proposition 4.3 with $\mathbb{Z}/2\mathbb{Z}$ replaced by \mathbb{C}^\times ,

$$0 \rightarrow H^1(\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)) \rightarrow H^2(G', \mathbb{C}^\times) \rightarrow H^2(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)^\mathbb{Z} \rightarrow 0.$$

This combined with the fact that second cohomology of a cyclic group with \mathbb{C}^\times -coefficients is zero, implies that

$$H^2(G', \mathbb{C}^\times) = H^1(\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)) = H^1(\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}).$$

Similarly

$$H^2(M^\times, \mathbb{C}^\times) = H^1(2\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)) = H^1(2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}).$$

We need to prove that the restriction map

$$H^1(\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times})[2] \cong \mathbb{Z}/2\mathbb{Z} \rightarrow H^1(2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times})[2] \cong \mathbb{Z}/2\mathbb{Z}$$

is injective. For this, consider the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

The above exact sequence gives rise to the following inflation-restriction exact sequence

$$0 \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}) \rightarrow H^1(\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}) \rightarrow H^1(2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}).$$

By the next lemma there is an isomorphism of $\widehat{\mathbb{F}_{q^2}^\times}$ with $\mathbb{F}_{q^2}^\times$ preserving the natural $\text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ action on these groups. Hence by Hilbert's theorem 90 we get that $H^1(\mathbb{Z}/2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}) = 0$. So the map

$$H^1(\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times}) \rightarrow H^1(2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times})$$

is injective and hence in particular on 2-torsions

$$H^1(\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times})[2] \rightarrow H^1(2\mathbb{Z}, \widehat{\mathbb{F}_{q^2}^\times})[2].$$

This proves that the map h is non-zero and an isomorphism. \square

Lemma 4.7. *There is an isomorphism of $\widehat{\mathbb{F}_{q^d}^\times}$ with $\mathbb{F}_{q^d}^\times$ such that the natural Galois action of $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ on $\widehat{\mathbb{F}_{q^d}^\times}$ becomes the inverse of the natural action of $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ on $\mathbb{F}_{q^d}^\times$ (where by “inverse” of an action of an abelian group G on a module M , we mean $g * m = (g^{-1})m$).*

Proof. Since the $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ operates by $x \mapsto x^q$ on $\mathbb{F}_{q^d}^\times$, the proof of the lemma is clear. \square

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, COLABA, MUMBAI 400005, INDIA

E-mail address: shiv@math.tifr.res.in